Higher-order Ising models on hypergraphs

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Why care about Ising models?

Originally: simplified model for ferromagnetisms

$$H = -h \sum_i s_i - J \sum_{\langle i,j \rangle} s_i s_j$$

- Spins that agree have a lower energy
- Heat disturbs this tendency

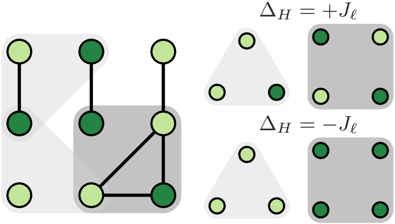
$\Longrightarrow$ one of the simplest models to show a **phase transition**.

More:

- Basic theory of cooperative phenomena.
- Mapping to NP-HARD problems (e.g. graph MAX-CUT).
- Pletora of applications for models with binary variables: neurons, votes, stock prices...
- ...

Higher-order Ising models[1]

$$H^{\mathrm{CS}} = -h\sum_i s_i - \sum_{\ell=1}^{\ell_{\max}} J_\ell \sum_{\{\sigma\in\mathcal{H}:|\sigma|=\ell\}} \left[2\bigotimes_{i\in\sigma} s_i - 1\right]$$



For $\ell_{\max} > 1$ different from traditional p-spin models:

$$H^{\mathrm{BS}} = -h\sum_i s_i - \sum_{\ell=1}^{\ell_{\max}} J_\ell \sum_{\{\sigma\in\mathcal{H}:|\sigma|=\ell\}} \prod_{i\in\sigma} s_i$$

[1] T. Robiglio, *et al.* **"Synergistic signatures of group mechanisms in higher-order systems."** Phys. Rev. Lett. 134, 137401 (2025).

Mean-field

Two main assumptions:

- Write the spin state at site i as:

$$s_i = \langle s_i \rangle + \Delta s_i = \langle s_i \rangle + (s_i - \langle s_i \rangle).$$

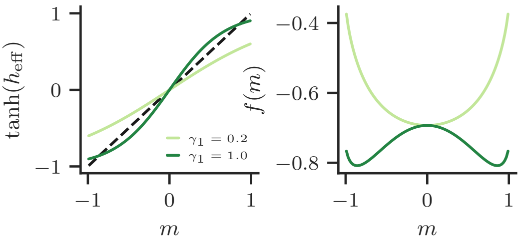
- Uniform expectation value in the system:

$$\langle s_i \rangle = m \;\; \forall i.$$

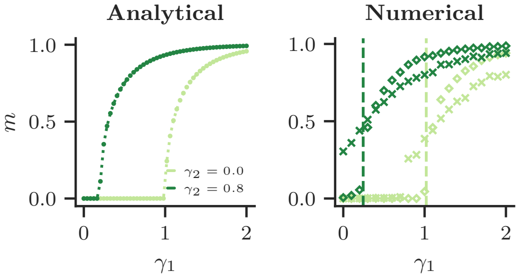
Fully decoupled Hamiltonian:

$$H(m) = -h_{\text{eff}} \sum_i s_i \implies m = \tanh \left[(\beta h_{\text{eff}}(m)) \right]$$

Results for $\ell_{\max} = 2$

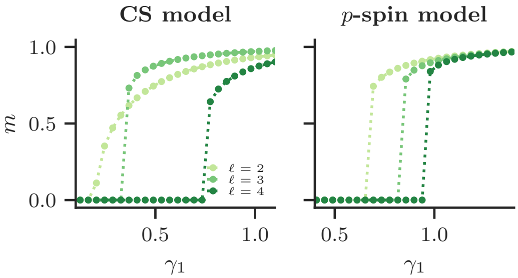


Continuos phase transition:



Beyond 3-body interactions

For $\ell_{\max} > 2$ we *"recover"* the abrupt phase[2] transition:



[2] I. Iacopini, *et al.* **"Simplicial models of social contagion."** Nat. Comm. 10, 2485 (2019).

Beyond mean-field (Georges-Yedidia expansion)

Magnetization dependent free-energy functional:

$$\mathcal{F}^{\beta}[\mathbf{m}] = \log \sum_{\{\mathbf{s}\}} \exp \left[-\beta H(\mathbf{s}) + \sum_i \rho_i^{\beta} (S_i - m_i) \right]$$

expanded around $\beta = 0$:

$$\mathcal{F}^{\beta}[\mathbf{m}] = \mathcal{F}^{\beta}[\mathbf{m}]\big|_{\beta=0} + \left. \frac{\partial \mathcal{F}^{\beta}[\mathbf{m}]}{\partial \beta} \right|_{\beta=0} \beta + \frac{1}{2} \left. \frac{\partial^2 \mathcal{F}^{\beta}[\mathbf{m}]}{\partial \beta^2} \right|_{\beta=0} \beta^2 + \ldots$$

At $\beta = 0$ spins are fully decoupled:

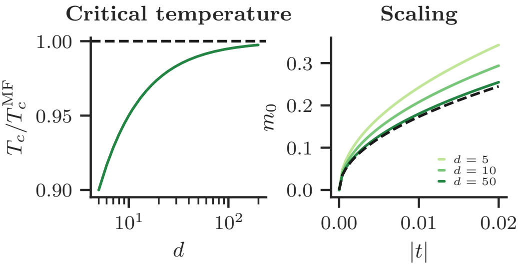
$$\langle \prod_{i=1}^{\alpha} s_i \rangle|_{\beta=0} = \prod_{i=1}^{\alpha} m_i$$

Beyond mean-field

$$\begin{array}{ccl} 0^{\text{th}} & \rightarrow & \text{Entropy} \\ 1^{\text{st}} & \rightarrow & \text{Mean-field energy} \\ \ldots & \rightarrow & \text{Corrections (e.g. Onsager reaction term)} \end{array}$$

On a d-regular 3-hypergraph:

$$f(m) = \log g(m) - \frac{\beta J_2 d}{2} m^2 - \frac{(\beta J_2)^2 d}{8} (1 - m^2)^2 + O(\beta^3)$$



Wrap-up

- There is a **new higher-order model** in town.
- **Don't stop at three body** when discussing higher-order model.
- Georges-Yedidia expansion to go **beyond mean-field**.

Related talk

Gangmin Son, Higher-order networks 1, Wed. 11:00 - 13:00

Thank you!

Check out the two papers:

Phys. Rev. Lett. 134, 137401
`arXiv:2411.19618`

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`skewed.de/lab`

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